(LIMIT OF A FUNCTION)

DEFINITION

f has a limit L at c, where $f: A \to \mathbb{R}$, $c \in A$

↓

f(x) are 'close to' L when x is 'close to' c.

↓

$\epsilon - \delta$ definition.

DEFINITION :- let (i) $A \subseteq \mathbb{R}$

(ii) c be a cluster point of A.

(iii) $f: A \to \mathbb{R}$.

So a real number L is said to be limit of f at c if, for given any $\epsilon > 0$ there exist ($\exists$) a $\delta > 0$ s.t. if $x \in A$ and $0 < |x - c| < \delta$, then

$$|f(x) - L| < \epsilon$$

Remark :- (1) It is not necessary that f is defined at c.

(2) $0 < |x - c| \Rightarrow x \neq c$

(3) $\delta(\epsilon)$

$L = \lim\limits_{x \to c} f(x)$ OR $L = \lim\limits_{x \to c} f$

OR $f(x) \to L$ as $x \to c$

Examples on ϵ-δ Definition

$\lim_{x \to c} f(x) = l$, if $|x-c| < \delta \Rightarrow |f(x)-L| < \epsilon$

Ex (1) $\lim_{x \to 2} (2x-3) = 1$

Proof: given, $|x-2| < \delta$ — (1)

Now $|f(x) - L|$

$= |2x-3-1|$

$= |2x-4|$

$= |2(x-2)|$

$= 2|x-2| < 2\delta$

$\Rightarrow |f(x)-L| < 2\delta$ — (2)

let $\epsilon = 2\delta$, then $\delta = \frac{\epsilon}{2}$ (we get)

Now by using (2)

$|f(x) - L| < \epsilon$. Hence Proof.

Ex (2) $\lim_{x \to a} \left(\frac{x^2 - a^2}{x-a}\right) = 2a$.

Proof given, $|x-a| < \delta$ — (1)

$|f(x)-L|$

$= \left|\left(\frac{x^2-a^2}{x-a}\right) - 2a\right|$

$= \left|\frac{(x+a)\cancel{(x-a)}}{\cancel{(x-a)}} - 2a\right|$

$= |x + a - 2a|$

$= |x - a| < \delta$

$\Rightarrow |f(x) - L| < \delta$

here let $\delta = \epsilon$

i.e. $|f(x) - L| < \delta$ Hence Proof.

Ex ③ $\lim_{x \to 2} x^2 = 4$

Proof :- given $|x - a| < \delta$

i.e. $|x - 2| < \delta$ — ①

Now, $|f(x) - L|$

$= |x^2 - 4|$

$= |(x+2)(x-2)|$ [by using formula $a^2 - b^2 = (a+b)(a-b)$

$= |x+2|\,|x-2|$ — ② i.e. $x^2 - 2^2 = (x+2)(x-$

[by $|ab| = |a|\,|b|$]

let, $\delta = 1$

$|x - 2| < \delta$

$|x - 2| < 1$

$\pm (x - 2) < 1$

$x - 2 < 1$ $-(x-2) < 1$

$\Rightarrow x < 3$ $x - 2 > -1$

$x > 1$

$\Rightarrow \quad 1 < x < 3$

then $\quad x < 3$

$x+2 < 3+2$

$x+2 < 5$

$$|f(x) - L| = |x+2|\,|x-2| < 5\delta$$

let $\epsilon = 5\delta$ and $\delta = \frac{\epsilon}{5}$

$\Rightarrow \quad |f(x) - L| < \epsilon \qquad$ Hence Proof

Ex (4.) Using $\epsilon-\delta$ definition of limit,

Prove, $\lim\limits_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$.

Proof We know that,

$\lim\limits_{x \to c} f(x) = L \rightarrow$ if $|x - L| < \delta \Rightarrow |f(x) - L| < \epsilon$

given, $|x - 0| < \delta$ — (1)

$|x| < \delta$

$|f(x) - L|$

$= \left|x \sin\left(\frac{1}{x}\right) - 0\right|$

$= \left|x \cdot \sin\left(\frac{1}{x}\right)\right|$ — (2)

$= |x|\,\left|\sin\left(\frac{1}{x}\right)\right|$ $\quad$ [by using Property $|a\,b| = |a|\,|b|$].

range of $\sin x$.

$$-1 \leq \sin x \leq 1$$

$$\Rightarrow \left|\sin\left(\tfrac{1}{x}\right)\right| < 1$$

then using eq^n 1 and 2

$$\left|x \cdot \sin\left(\tfrac{1}{x}\right)\right|$$

$$= |x| \left|\sin\left(\tfrac{1}{x}\right)\right|$$

$$< |x| (1)$$

$$< \delta$$

let $\delta = \epsilon$

then $|f(x) - L| < \epsilon$ Hence Proof